

ON STATISTICAL ARC LENGTH OF THE RIEMANN $Z(t)$ -CURVE

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ABSTRACT. In this paper we study certain stochastic process that is generated by the Riemann-Siegel formula. Further, we construct corresponding statistical model by a way similar to those used in telecommunication. We define statistical arc length of the Riemann $Z(t)$ -curve in this model and obtain an asymptotic formula for that length. This paper is English remake of our work of reference [4].

1. INTRODUCTION

1.1. In the paper [5] we have studied the following integral

$$(1.1) \quad \int_T^{T+H} \sqrt{1 + \{Z'(t)\}^2} dt = L(T, H) = L,$$

i.e. the arc length of the Riemann curve

$$y = Z(t), \quad t \in [T, T + H], \quad T \rightarrow \infty,$$

where (see [7], pp. 79, 329)

$$Z(t) = e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + it\right),$$

$$\vartheta(t) = -\frac{t}{2} \ln \pi + \operatorname{Im} \ln \Gamma\left(\frac{1}{4} + i\frac{t}{2}\right).$$

Next, we will denote the roots of the equations

$$Z(t) = 0, \quad Z'(t) = 0$$

by the symbols

$$\{\gamma\}, \quad \{t_0\}; \quad t_0 \neq \gamma$$

correspondingly.

Remark 1. On the Riemann hypothesis, the points of the sequences $\{\gamma\}$ and $\{t_0\}$ are separated each from other (see [2], Corollary 3), i.e. in this case we have

$$\gamma' < t_0 < \gamma'',$$

where γ', γ'' are neighboring points of the sequence $\{\gamma\}$. Of course, $Z(t_0)$ is an extremal value of the function $Z(t)$ in some neighborhood of the point t_0 .

Namely, we have proved (see [5]) for integral (1.1) the following formula

$$(1.2) \quad \int_T^{T+H} \sqrt{1 + \{Z'(t)\}^2} dt = 2 \sum_{T \leq t_0 \leq T+H} |Z(t_0)| + \Theta H + \mathcal{O}\left(T^{\frac{A}{\ln T}}\right),$$

$$T \rightarrow \infty, \quad \Theta = \Theta(T, H) \in (0, 1), \quad H = T^\epsilon$$

for every fixed $\epsilon > 0$.

The proof of the formula (1.2) was based on the following

$$(1.3) \quad Z'(t) = -2 \sum_{n < P} \frac{1}{\sqrt{n}} \ln \frac{P}{n} \sin\{\vartheta(t) - t \ln n\} +$$

$$+ \mathcal{O}(T^{-1/4} \ln T), \quad t \in [T, T+U], \quad U \in (0, \sqrt{T}],$$

that was obtained in our paper [3] as a variant of the Riemann-Siegel formula. One sight on formulae (1.1)–(1.3) is sufficient for insight of the difficulty of the problem about the asymptotic formula of the arc length of the Riemann curve.

1.2. In connection with formulae (1.1)–(1.3) we give the following

Remark 2. The main difficulty lies probably in that we are absent of something like famous Dirac procedure

$$\sqrt{m^2 c^2 + p_1^2 + p_2^2 + p_3^2} = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta,$$

i.e. extracting of the square root of the operator - see [1], p. 255, equations (5), (7). This was the way towards Dirac's relativistic wave equation for the electron - the fundamental law of the quantum mechanics.

1.3. Next, let us remind, in connection with (1.1)–(1.3), that in our recent paper we have proved the following result on metamorphoses: there is an infinite set of elements

$$(1.4) \quad \{\alpha_0(T), \alpha_1(T), \dots, \alpha_k(T)\}, \quad T \in (T_0, +\infty), \quad T_0 > 0$$

where T_0 is sufficiently big, such that

$$(1.5) \quad \prod_{r=1}^k \left| \sum_{n \leq \tau(\alpha_r)} \frac{2}{\sqrt{n}} \cos\{\vartheta(\alpha_r) - \alpha_r \ln n\} + \mathcal{O}(\alpha_r^{-1/4}) \right| \sim$$

$$\sim \sqrt{\frac{\Lambda}{\left| \sum_{n \leq \tau(\alpha_0)} \frac{2}{\sqrt{n}} \cos\{\vartheta(\alpha_0) - \alpha_0 \ln n\} + \mathcal{O}(\alpha_0^{-1/4}) \right|}},$$

$$T \rightarrow \infty,$$

where

$$\Lambda = \sqrt{2\pi} \frac{H}{H_k} \ln^k T, \quad \tau(t) = \sqrt{\frac{t}{2\pi}}, \quad k = 1, \dots, k_0, \quad k_0 \in \mathbb{N},$$

i.e. to the infinite subset

$$\{\alpha_1(T), \dots, \alpha_k(T)\}$$

an infinite set of metamorphoses of the multiform on the left-hand side of (1.5) into quite distinct form on the right-hand side of (1.5) corresponds.

Now we rewrite the formula (1.5) in the following form

$$(1.6) \quad \sqrt{\frac{1}{\Lambda} \left| \sum_{n \leq \tau(\alpha_0)} \frac{2}{\sqrt{n}} \cos\{\vartheta(\alpha_0) - \alpha_0 \ln n\} + \mathcal{O}(\alpha_0^{-1/4}) \right|} \sim \prod_{r=1}^k \left| \sum_{n \leq \tau(\alpha_r)} \frac{2}{\sqrt{n}} \cos\{\vartheta(\alpha_r) - \alpha_r \ln n\} + \mathcal{O}(\alpha_r^{-1/4}) \right|^{-1}.$$

Remark 3. We see, in connection with (1.1)–(1.3) and the Remark 2, that the square root of the weighted nonlinear monoform on the left-hand side of (1.6) is asymptotically expressed as the rational function of the multiform for infinite set of elements (1.4).

1.4. In this paper, we introduce new method to study the main integral (1.1). Namely, we use certain stochastic process generated by the formula (1.3). We use simple properties of this process together with the Ljapunov's central limit theorem for a construction of an integral that we call *statistical arc length of the Riemann $Z(t)$ -curve*. Consequently, an asymptotic formula for this integral is obtained.

2. TRANSFORMATION OF THE FORMULA (1.1)

In the formula (see (1.1))

$$L = L(T, U) = \int_T^{T+U} \sqrt{1 + \{Z'(t)\}^2} dt$$

we put

$$Z'(t) = Z_1(t) + R_1(t), \quad t \in [T, T + U],$$

where (see (1.3))

$$Z_1(t) = 2 \sum_{n < P} \frac{1}{\sqrt{n}} \ln \frac{P}{n} \cos\{\vartheta(t) - t \ln n + \pi/2\},$$

$$R_1(t) = \mathcal{O}(T^{-1/4} \ln T).$$

Since

$$\ln \frac{P}{n} = \mathcal{O}(\ln P), \quad \ln \frac{P}{n-1} > \ln \frac{P}{n}, \quad 2 \leq n < P,$$

then, similarly to [7], pp. 92, 93, we obtain the following estimate

$$Z_1(t) = \mathcal{O}(T^{1/6} \ln^2 T), \quad t \in [T, T + U].$$

Hence,

$$\{Z'(t)\}^2 = \{Z_1(t)\}^2 + \mathcal{O}(T^{-1/12} \ln^3 T),$$

and

$$\begin{aligned} \sqrt{1 + \{Z'(t)\}^2} &= \sqrt{1 + \{Z_1(t)\}^2} \left\{ 1 + \mathcal{O} \left(\frac{T^{-1/12} \ln^3 T}{1 + \{Z_1(t)\}^2} \right) \right\} = \\ &= \sqrt{1 + \{Z_1(t)\}^2} + \mathcal{O}(T^{-1/12} \ln^3 T). \end{aligned}$$

Consequently, we have the following

Lemma 1.

$$(2.1) \quad L = L(T, U) = \int_T^{T+U} \sqrt{1 + \{Z_1(t)\}^2} dt + \mathcal{O}(T^{-1/12} \ln^3 T),$$

$$(2.2) \quad Z_1(t) = 2 \sum_{n < P} \frac{1}{\sqrt{n}} \ln \frac{P}{n} \cos\{\vartheta(t) - t \ln n + \pi/2\}, \quad U \in (0, \sqrt{T}].$$

3. DEFINITION OF THE STATISTICAL ARC LENGTH OF THE RIEMANN CURVE AND THEOREM

3.1. Let

$$\varphi_n \in [-\pi, \pi], \quad n < P$$

be the system of independent random variables each of them uniformly distributed on the segment $[-\pi, \pi]$. Putting these φ_n into the arguments of cosine-functions in (2.2) one obtains the following stochastic process

$$(3.1) \quad \begin{aligned} \Phi_1(t) &= \Phi_1(t, \varphi_1, \varphi_2, \dots) = \\ &= 2 \sum_{n < P} \frac{1}{\sqrt{n}} \ln \frac{P}{n} \cos\{a(t, n) + \varphi_n\}, \end{aligned}$$

where

$$\begin{aligned} a(t, n) &= \vartheta(t) - t \ln n + \frac{\pi}{2}, \\ t &\in [T, T + U], \quad U \in (0, \sqrt{T}]. \end{aligned}$$

Remark 4. The following is true: every realization

$$\Phi_1(t, \bar{\varphi}_1, \bar{\varphi}_2, \dots), \quad t \in [T, T + U]$$

of the stochastic process (3.1) is a continuous function of the variable t for every admissible vector

$$\bar{\varphi} = (\bar{\varphi}_1, \bar{\varphi}_2, \dots)$$

and, consequently, there is the Riemann integral

$$\int_T^{T+U} \Phi_1(t, \bar{\varphi}) dt, \quad \forall \bar{\varphi}.$$

We also use the one-dimensional Gaussian density of probability

$$(3.2) \quad w_\infty(\Phi_1) = \frac{1}{2} \frac{1}{\ln^{3/2} P} e^{-\beta(\Phi_1)^2}, \quad \beta = \frac{3}{4 \ln^3 P}.$$

3.2. Now, we define the following stochastic process

$$(3.3) \quad \Phi_2 = \Phi_2(T, U, \varphi) = \int_T^{T+U} \sqrt{1 + \{\Phi_1(t)\}^2} dt, \quad U \in (0, \sqrt{T}],$$

for every sufficiently big T . Expectation value of this process is given by the formula

$$(3.4) \quad E(\Phi_2) = \int_T^{T+U} E(\sqrt{1 + (\Phi_1)^2}) dt.$$

Next, we define the following asymptotic expectations

$$(3.5) \quad E_\infty(\sqrt{1 + (\Phi_1)^2}) = \int_{-\infty}^{\infty} \sqrt{1 + (\Phi_1)^2} w_\infty(\Phi_1) d\Phi_1,$$

and, consequently, (see (3.3)–(3.5))

$$(3.6) \quad E_\infty(\Phi_2) = \int_T^{T+U} E_\infty(\sqrt{1 + (\Phi_1)^2}) dt.$$

Thus, the comparison of the formulae (2.2), (3.1) and (2.1), (3.6) leads us to the following

Definition.

$$(3.7) \quad \langle L(t, U) \rangle|_S = E_\infty(\Phi_2),$$

where

$$\langle L(t, U) \rangle|_S$$

is the statistical arc length of the Riemann $Z(t)$ -curve.

Consequently, we notice that the basis of constructed statistical model lies in the following (purely) mathematical

Theorem.

$$(3.8) \quad E_\infty(\sqrt{1 + (\Phi_1)^2}) \sim \frac{1}{\sqrt{6\pi}} \ln^{3/2} T, \quad T \rightarrow \infty.$$

Since (see (3.6), (3.8))

$$(3.9) \quad E_\infty(\Phi_2) \sim \frac{1}{\sqrt{6\pi}} U \ln^{3/2} T, \quad T \rightarrow \infty,$$

then we obtain from (3.9) the following

Corollary.

$$\langle L(t, U) \rangle|_S \sim \frac{1}{\sqrt{6\pi}} U \ln^{3/2} T, \quad U \in (0, \sqrt{T}], \quad T \rightarrow \infty.$$

4. STATISTICAL CONSIDERATIONS ABOUT THE GAUSSIAN DISTRIBUTION

4.1. The following lemma holds true

Lemma 2.

$$(4.1) \quad E(\Phi_1) = 0, \quad \text{Var}(\Phi_1) \sim \frac{2}{3} \ln^3 P, \quad T \rightarrow \infty.$$

Proof. We have (see (3.1))

$$\Phi_1(t; \varphi) = \sum_{n < P} X_n, \quad X_n = \frac{2}{\sqrt{n}} \ln \frac{P}{n} \cos\{\varphi_n + a\}.$$

Since φ_n are independent and uniformly distributed then we obtain that

$$E(X_n) = 0, \quad \text{Var}(X_n) = \frac{2}{n} \ln^2 \frac{P}{n}$$

and

$$E(\Phi_1) = 0, \quad \text{Var}(\Phi_1) = 2 \sum_{n < P} \frac{1}{n} \ln^2 \frac{P}{n}.$$

Next, for Var we use the Euler-MacLaurin summation formula (comp. [7], p. 13) for the function

$$f(x) = \frac{1}{x} \ln^2 \frac{P}{x}, \quad x \in [1, P].$$

Since

$$f'(x) = \mathcal{O}\left(\frac{\ln^2 P}{x^2}\right), \quad f(1) = \mathcal{O}(\ln^2 P), \quad f(P) = 0,$$

then

$$\begin{aligned} \sum_{n < P} f(n) &= \int_1^P \ln^2 \frac{P}{x} \frac{dx}{x} + \mathcal{O}\left(\ln^2 P \int_1^P \frac{dx}{x^2}\right) + \mathcal{O}(\ln^2 P) = \\ &= \frac{1}{3} \ln^3 P + \mathcal{O}(\ln^2 P), \end{aligned}$$

and, consequently,

$$\text{Var}(\Phi_1) \sim \frac{2}{3} \ln^3 P, \quad T \rightarrow \infty.$$

□

4.2. Next, we give the following

Lemma 3. It is true that in our case the Ljapunov's condition in the central limit theorem is fulfilled.

Proof. Since

$$\begin{aligned} E(|X_n - E(X_n)|^3) &= E(|X_n|^3) = \frac{8}{n^{3/2}} \ln^3 P \frac{1}{2\pi} \int_{-\pi}^{\pi} |\cos^3(\varphi_n + a)| d\varphi_n < \\ &< \frac{8}{n^{3/2}} \ln^3 \frac{P}{n}, \end{aligned}$$

then we have the following estimate for the third absolute central moment

$$\sum_{n < P} E(|X_n|^3) \leq 8 \sum_{n < P} \frac{1}{n^{3/2}} \ln^3 \frac{P}{n} < 8 \ln^3 P \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} = A \ln^3 P.$$

Now, (see (4.1))

$$B_P = \sqrt{\text{Var}(\Phi_1)} \sim \sqrt{\frac{2}{3}} \ln^{3/2} P$$

and, consequently,

$$(4.2) \quad \frac{1}{B_P^3} \sum_{n < P} E(|X_n|^3) < 2A \frac{\ln^3 P}{\ln^{9/2} P} \xrightarrow{T \rightarrow \infty} 0,$$

i.e. the Ljapunov's condition is fulfilled. □

Remark 5. Our choice of the Gaussian asymptotics in (3.2) is based upon the results (4.1) and (4.2), respectively.

5. PROOF OF THEOREM

We put (see (3.2) – (3.5))

$$(5.1) \quad E_{\infty}(\sqrt{1 + (\Phi_1)^2}) = \sqrt{\frac{3}{\pi}} \frac{1}{\ln^{3/2} P} F(\beta),$$

where

$$(5.2) \quad F(\beta) = \int_0^{\infty} \sqrt{1 + x^2} e^{-\beta x^2} dx, \quad \beta = \frac{3}{4 \ln^3 P}.$$

We need to obtain an asymptotic formula for $F(\beta)$ for small values of β (i.e. for large values of T).

First of all, we have that

$$\begin{aligned} F(\beta) &= \int_0^\infty \cosh^2 t e^{-\beta \sinh^2 t} dt = \frac{1}{2} e^{\beta/2} \int_0^\infty (\cosh 2t + 1) e^{-\frac{\beta}{2} \cosh 2t} dt = \\ &= \frac{1}{4} e^{\beta/2} \int_0^\infty (\cosh t + 1) e^{-\frac{\beta}{2} \cosh t} dt. \end{aligned}$$

Now, we use the Schläfli's integral (see [8], p. 81)

$$\int_0^\infty e^{-z \cosh t} \cosh \nu t dt = K_\nu(t), \quad \operatorname{Re}\{z\} > 0,$$

for the modified Bessel's function of the second kind, and then we obtain

$$(5.3) \quad F(\beta) = \frac{1}{4} e^{\beta/2} \left\{ K_0\left(\frac{\beta}{2}\right) + K_1\left(\frac{\beta}{2}\right) \right\}.$$

Further, we use the following representations of the K_ν -functions (see [8], p. 80)

$$\begin{aligned} K_0(z) &= I_0(z) \ln \frac{z}{2} + \sum_{m=1}^\infty \frac{z^{2m}}{2^m (m!)^2} \psi(m+1), \\ K_n(z) &= \frac{1}{2} \sum_{m=0}^{n-1} (-1)^m \frac{(n-m-1)!}{m! \left(\frac{z}{2}\right)^{n-2m}} + \\ &+ (-1)^{n+1} \sum_{m=0}^\infty \frac{\left(\frac{z}{2}\right)^{n+2m}}{m! (n+m)!} \left\{ \ln \frac{z}{2} - \frac{1}{2} \psi(m+1) - \frac{1}{2} \psi(n+m+1) \right\}, \end{aligned}$$

where (see [8], p. 77)

$$I_0(z) = \sum_{m=0}^\infty \frac{1}{(m!)^2} \left(\frac{z}{2}\right)^{2m},$$

and (see [9], p. 241)

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = -c - \sum_{k=0}^\infty \left(\frac{1}{k+x} - \frac{1}{k+1} \right),$$

where c is the Euler's constant.

Since

$$\psi(m+1), \psi(n+m+1) = \mathcal{O}(1),$$

and (see (5.2))

$$\begin{aligned} \frac{\beta}{2} &= \frac{3}{8 \ln^3 P}, \quad \ln \frac{\beta}{4} = -3 \ln \ln P + \mathcal{O}(1), \\ I_0\left(\frac{\beta}{2}\right) &= 1 + \mathcal{O}\left(\frac{1}{\ln^6 P}\right), \end{aligned}$$

then

$$\begin{aligned} K_0\left(\frac{\beta}{2}\right) &= 3 \ln \ln P + \mathcal{O}(1), \\ K_1\left(\frac{\beta}{2}\right) &= \frac{2}{\beta} + \mathcal{O}(\beta |\ln \beta|) = \frac{8}{3} \ln^3 P + \mathcal{O}\left(\frac{\ln \ln P}{\ln^3 P}\right). \end{aligned}$$

Consequently (see (5.3))

$$F(\beta) = \frac{2}{3} \ln^3 P + \mathcal{O}(\ln \ln P),$$

and from this (see (5.1)) the formula (3.8) follows, of course,

$$\ln^{3/2} P \sim \frac{1}{2\sqrt{2}} \ln^{3/2} T, \quad T \rightarrow \infty.$$

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REFERENCES

- [1] P.A.M. Dirac, 'The principles of quantum mechanics', Oxford at the Clarendon Press, (1958).
- [2] J. Moser, 'Some properties of the Riemann zeta-function on the critical line', Acta Arith., 26, (1974), 33-39, (in Russian), arXiv: 0710.0943. (1995).
- [3] J. Moser, 'On the roots of the equation $Z'(t) = 0$ ', Acta. Arith., 40, (1981), 97-107, (in Russian), arXiv: 1303.0967.
- [4] J. Moser, 'Ljapunov theorem and the statistical length of a curve in the theory of Riemann-zeta function', Acta Math. Univ. Comen., 54-55 (1988), 147-163.
- [5] J. Moser, 'Riemann hypothesis and the arc length of the Riemann $Z(t)$ -curve', arXiv: 1404.1717.
- [6] J. Moser, 'Jacob's ladders, ζ -factorization and infinite set of metamorphoses of a multiform', arXiv: 1501.07705.
- [7] E. C. Titchmarsh, 'The theory of the Riemann zeta-function', Clarendon Press, Oxford, 1951.
- [8] G.N. Watson, *A treatise on the theory of Bessel-functions*, Cambridge at the University Press, 1922.
- [9] E.T. Whittaker, G.N. Watson, *A course of modern analysis*, Cambridge, University Press, 1927.

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